The stationary KdV hierarchy and $so(2,1)$ as a spectrum generating algebra

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Abstract

The family $\mathcal{F}_L$ of all potentials $V(x)$ for which the Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$ in one space dimension possesses a high order Lie symmetry is determined. A sub-family $\mathcal{F}^2_{SGA}$ of $\mathcal{F}_L$, which contains a class of potentials allowing a realization of $so(2,1)$ as spectrum generating algebra of H through differential operators of finite order, is identified. Furthermore and surprisingly, the families $\mathcal{F}^2_{SGA}$ and $\mathcal{F}_L$ are shown to be related to the stationary KdV hierarchy. Hence, the 'harmless' Hamiltonian H connects different mathematical objects, high order Lie symmetry, realization of $so(2,1)$-spectrum generating algebra and families of nonlinear differential equations. We describe in a physical context the interplay between these objects.

I. Motivation and Background

In the two internal reports for the International Center of Theoretical Physics (ICTP) [1, 2] written in the early seventies, a complete classification of sym-

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metric Hamiltonians in one space dimension on $L^2(\mathbf{R}_x^1, dx)$

$$H = \gamma \frac{d^2}{dx^2} + V(x), \quad \gamma < 0 \tag{1}$$

having the Lie algebra $so(2, 1)$ as a 'spectrum generating algebra' (SGA) has been obtained. This result has been published only recently in connection with a Lecture on Memory of A.O. Barut [3]. In [1, 2] the following definition of SGA is used: a differential operator A of the order n' has a spectrum generating (Lie) algebra L with generators g_i, $(i = 1, ..., m, \ m = \dim L)$ if there exist a realization R of L through differential operators of an order $n \geq n'$, such that

$$A = \sum_{i=0}^{m} \alpha_i R(g_i), \quad \alpha_i \in \mathbf{R}. \tag{2}$$

The Hamiltonian (1) is a differential operator on $L^2(\mathbf{R}_x^1, dx)$ with $n' = 2$. A realization R of $so(2, 1)$ with standard basis $\mathcal{L}$ spanned by g_1, g_2, g_3 through nth order differential operators on a suitable complex function space over x reads

$$R(g_i) = \sum_{j=0}^{n} a_{ij}(x) \frac{d^j}{dx^j}, \quad i = 1, 2, 3, \tag{3}$$

where $a_{ij}(x)$ are complex functions such that the commutation relations

$$[R(g_1), \ R(g_2)] = -R(g_3), \quad [R(g_2), \ R(g_3)] = R(g_1),$$
$$[R(g_3), \ R(g_1)] = -R(g_2) \tag{4}$$

are fulfilled. So $R(so(2, 1))$ is a SGA of (1) if there exist constants $\alpha_i \in \mathbf{R}$, $(i = 1, 2, 3)$ such that the equality

$$H \equiv \gamma \frac{d^2}{d^2 x} + V(x) = \sum_{j=1}^{3} \alpha_j R(g_j) \tag{5}$$

holds.

The relations (3)–(5) impose restrictions both on the coefficients $a_{ij}(x)$ and on the potential $V(x)$. Solving these restrictions we find two different families $\mathcal{F}_{SGA} = \left\{ \mathcal{F}_1^{(2)}, \ \mathcal{F}_2^{(n)}, \ n \geq 2 \right\}$ of those potentials which allow $so(2, 1)$ as SGA for (1). Now, we can use properties of the representation of $so(2, 1)$ through $R(so(2, 1))$ in order to calculate the spectrum of H, if $R(so(2, 1))$

acts, e.g., on $L^2(\mathbf{R}_x^1, dx)$. If, furthermore, $R(g_i)$ are essentially self-adjoint in a common dense domain and if the representation is integrable, then we can use the known theory for unitary representations of $so(2,1)$. This is the background of the term 'spectrum-generating algebra' as suggested in [4, 5]. There were many results in this field for different Hamiltonians, Lie algebras and physical systems (see, e.g., the recent review [6]) but no general study in the sense of [1, 2].

The motivation of the present paper is to show (in Section III) that $\mathcal{F}_2^{(n)} \subset \mathcal{F}_{SGA}$ can be read from the stationary KdV hierarchy and that this surprising connection between the KdV hierarchy and $so(2,1)$ has its origin in a certain higher order Lie symmetry of the corresponding Hamiltonian (5). In Section II we sketch the results of [1, 2] on which our discussion are based.

II. On Some Known Results

To classify those $V(x)$ which are solutions of (3)–(5) we reduce the calculation to special choices of parameters $\alpha_1, \alpha_2, \alpha_3$ through transformations of the standard basis $\mathcal{M}$ (which is not unique) to another standard basis $\mathcal{M}'$. As a result, we reduce the problem to the following two cases ($\lambda \in \mathbf{R}, \lambda \neq 0$):

$$\text{Case 1.} \quad \alpha_1^2 + \alpha_2^2 \neq \alpha_3^2 \quad \text{with} \quad H = \lambda R(g_i), \; i = 1, 2, \qquad (6)$$

$$\text{Case 2.} \quad \alpha_1^2 + \alpha_2^2 = \alpha_3^2 \quad \text{with} \quad H = \lambda(R(g_1) + R(g_3)). \qquad (7)$$

Case 2 is denoted as the 'light cone case' . In both cases (3)–(5) lead for a fixed n to set of coupled differential equations of the order n for $a_{ij}(x)$, ($i = 1, 2, 3$, $j = 1, \ldots, n$) and $V(x)$. We assume that $R(g_i)$ are symmetric operators and that $V(x)$ is a real function. A clumsy but straightforward calculations shows that in Case 1 a solution exists only for $n = 2$ with a family $\mathcal{F}_{SGA}^1$ of corresponding potentials

$$\mathcal{F}_{SGA}^1(\lambda_1, \lambda_2, C) = \{V(x) \mid V(x) = \lambda_1(x - C)^2 + \lambda_2(x - C)^{-2},$$

$$\lambda_1, \lambda_2, c \in \mathbf{R}, \lambda_1 \neq 0\}. \qquad (8)$$

In Case 2 a solution exists for all $n \geq 2$. The corresponding family $\mathcal{F}_{SGA}^2$ consists of potentials that are solutions of nonlinear differential equation

$$\mathcal{F}_{SGA}^2 = \left\{ V(x) \mid -\left(\frac{x}{2}V' + V\right) + \sum_{j=0}^{N-1} C_j F_j + F_N = 0 \right\}, \qquad (9)$$

3

where $N = \left[\frac{n-1}{2}\right]$, $n \geq 2$; F_i are some polynomials of $V(x)$ and its derivatives up to the order $2i+1$ $(i = 0, 1, \ldots, N)$ which will be derived later in another context (see, below (12), (13)); $C_0, C_1, \ldots, C_{N-1}$ are arbitrary real constants.

The family $\mathcal{F}^2_{SGA}$ has a peculiar structure, the facts that the equations are equal for $n = 2N+1$ and $n = 2N+2$, $N = 1, 2, \ldots$ and the equation for $n = n_1 > n_2$ contains all terms of the equation for $n = n_2$ are two of these peculiarities. This structure was not elucidated in $[1, 2]$[1]. Relations to other mathematical notions and objects were not found. The present paper fills this gap.

In order to simplify the following calculations we scale the variable x and thus get $\gamma = -1$. So the Schrödinger operator (1) takes the form

$$H = -\frac{d^2}{dx^2} + V(x). \tag{10}$$

III. SGA, Lie symmetries and KdV hierarchy

A. Aim and Strategy

We will show that the family $\mathcal{F}^2_{SGA}$ (and the the above mentioned peculiarities) are connected with a high order Lie symmetry Q of the stationary Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = 0 \tag{11}$$

with $[Q, H] = \kappa H$ and that the coefficients F_i, $(i = 0, 1, \ldots, N)$ of equation (9) appear, surprisingly, in the stationary KdV hierarchy

$$\sum_{j=0}^{N-1} C_j F_j + F_N = 0.$$

We remind that the stationary KdV hierarchy is obtained successively by the repeated action of the integro-differential operator (second recursive operator

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[1]This was the reason why the authors of $[1, 2]$ decided to present the results as internal reports.

for the KdV equation [7]–[9])

$$\mathcal{R} = -\frac{1}{4}\frac{d^2}{dx^2} - V(x) - \frac{1}{2}V'(x)\left(\frac{d}{dx}\right)^{-1} \qquad (12)$$

generating $F_i, (i = 0, 1, \ldots)$ through

$$F_{i+1} = \mathcal{R}\, F_i, \quad i = 0, 1, \ldots \quad \text{with } F_0 = -\frac{1}{2}V'(x). \qquad (13)$$

To get $so(2,1)$ as SGA for H, the Lie symmetry has to be specified through $\kappa = -1$.

Our strategy is the following. We construct at the first step the family of all potentials $V(x)$ for which $H = -\frac{d^2}{dx^2} + V(x)$ commutes with an nth order differential operator $Q \in \mathcal{L}$, i.e., the family of potentials with an nth order Lie symmetry provided $\kappa = 0$ (Theorem 1). At the second step, we generalize this result for an arbitrary κ (Theorem 2), hence we get the special case $\kappa = -1$ yielding SGA for the Schrödinger operator.

B. Lie symmetries

First, we remind that an n-th order differential operator

$$Q = \sum_{j=0}^{n} q_j(x)\frac{d^j}{dx^j}, \quad \frac{d^0}{dx^0} \overset{\text{def}}{=} 1 \qquad (14)$$

is called an n-th order (Lie) symmetry operator of the Schrödinger equation (11) if it transforms the set of its solutions into itself. An equivalent (and more algorithmic) definition is the following one [10]. The operator Q is a symmetry operator of equation (11) if there is such an m-th order differential operator P that

$$[Q,\ H] = PH. \qquad (15)$$

Evidently, if Q is a symmetry operator of an equation $H\psi = 0$, then an operator

$$\tilde{Q} = Q + \sum_{j=0}^{N} \gamma_j H^j$$

with arbitrary constant γ_j is also a symmetry operator. Given a symmetry operator Q, the operator $\tilde{Q}$ gives no new information on the structure of a

5

set of solutions of the equation under study. That is why, it is excluded from further considerations. In addition, we impose on the symmetry operators in (15), as an additional constraint, the condition $P = \kappa \, \mathbf{1}$, where κ is a real constant.

Hence, it is only necessary to consider representatives of the equivalence classes of the quotient of the linear space of differential operators (14) satisfying the equation

$$[Q, \ H] = \kappa H, \quad \kappa = \text{const} \tag{16}$$

with respect to the equivalence relation

$$Q_1 \sim Q_2 \quad \text{if} \quad Q_1 - Q_2 = \sum_{j=0}^{N} \gamma_j H^j$$

with some $N \in \mathbf{N}$ and constant γ_j. We denote this quotient space as $\mathcal{L}$.

To apply this notion to our problem we note that in the light cone case 2 due to the commutation relations of the algebra $so(2,1)$ the following equality

$$[R(g_2), \ H] = -H \tag{17}$$

is fulfilled. Hence $R(g_2) = Q$ is a high order Lie symmetry of equation (11) for the light cone case with $P = -1$, i.e. with $\kappa = -1$. We add that in the regular case (6) the algebra $so(2,1)$ does not contain a symmetry operator of the corresponding Schrödinger equation.

Thus a high-order Lie symmetry of the Schrödinger equation (11) is 'responsible' for restricting the potential $V(x)$. Furthermore, a spectrum generating algebra $so(2,1)$ for H in (1) can be constructed.

Before formulating the principal assertions we prove an auxiliary lemma.

Lemma 1 *The Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$ commutes with an n-th order differential operator $Q \in \mathcal{L}$ (14) if and only if the Schrödinger equation*

$$\left(-\frac{d^2}{dx^2} + V(x) + \varepsilon\right)\psi = 0, \quad \varepsilon \neq 0 \tag{18}$$

admits a Lie symmetry of the form

$$\hat{Q} = a(x, \varepsilon)\frac{d}{dx} + b(x, \varepsilon) \equiv \left(\sum_{j=0}^{N} a_j(x)\varepsilon^j\right)\frac{d}{dx} + \sum_{j=0}^{N} b_j(x)\varepsilon^j, \tag{19}$$

where $N = \left[\frac{n-1}{2}\right]$, $a_N = 1$ and ε is a continuous real parameter.

6

We give a sketch of the proof omitting technical details. To show that (18) implies (19) we compute the commutator on the left-hand side of the equality $[Q, H] = 0$ and equate coefficients of the operator $\frac{d^{n+1}}{dx^{n+1}}$. Hence we get in (14), $q_n = $ const. Consequently, without loosing generality we may choose $q_n = 1$ and look for a symmetry operator Q in the form

$$Q = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} q_j(x)\frac{d^j}{dx^j}. \tag{20}$$

Furthermore, as Q belongs to $\mathcal{L}$, the number n is odd and can be represented as $n = 2N + 1$ with some $N \in \mathbf{N}$.

As Q commutes with H, it commutes with a shifted Hamiltonian $H + \varepsilon\mathbf{1}$ with an arbitrary constant ε as well. Thus Q is a symmetry operator of (18).

Next, we make use of a well-known fact in the theory of high-order Lie symmetries of linear differential equations (see, e.g., [10]). Let

$$X = \sum_{j=0}^{n} q_j(x)\frac{d^j}{dx_j}$$

be a symmetry operator of the linear differential equation $A\psi = 0$ and A, R be differential operators of finite order. Then, $\tilde{X} = X + RH$ is also a symmetry operator of the equation $H\psi = 0$. Choosing the operator R properly we can cancel in Q all powers of the operator $\frac{d}{dx}$ of the degree $k > 1$. According to the last remark the first-order operator $\hat{Q}$ obtained in this way is still a symmetry operator of equation (18). Consequently, if the Schrödinger equation (18) admits a high order symmetry operator, then it necessarily admits a first-order Lie symmetry. The latter can be easily shown to have the form (19).

Suppose now that (18) admits a first-order Lie symmetry of the form (19). Then we can cancel all the powers of ε by adding an appropriately chosen polynomial of H with variable coefficients. As a result, we get $(2N + 1)$th order differential operator which is still a symmetry of the equation under study. Moreover, it is straightforward to verify that this operator commutes with H which is the same as what was to be proved. $\triangleright$

Using this result, we conclude that the problem of description of $(2N + 1)$-th order operators $Q \in \mathcal{L}$ commuting with the Schrödinger operator H is equivalent to the study of its usual Lie symmetry given by (19). This

remarkable fact connects SGA in the light cone case to the stationary KdV hierarchy.

Theorem 1 *The Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$ commutes with an n-th order differential operator $Q \in \mathcal{L}$ if and only if the potential $V(x)$ satisfies of the following families $\mathcal{F}_L^0$ of nonlinear differential equations*

$$\mathcal{F}_L^0 = \left\{ V(x) \mid G(V) \equiv \sum_{j=0}^{N-1} C_j F_j + F_N = 0, \ N \in \mathbf{N} \right\}, \qquad (21)$$

where F_i are polynomials in $V(x)$ and its derivatives forming the stationary KdV hierarchy (13), $C_0, C_1, \ldots, C_{N-1}$ are some real constants.

Proof. The proof is simplified substantially if we use Lemma 1 for $H + \varepsilon \mathbf{1}$, because this gives a possibility to use the well-known technique of the soliton theory.

Inserting (18), (19) into the invariance condition $[H + \varepsilon \mathbf{1}, \hat{Q}] = R(H + \varepsilon \mathbf{1})$ we get

$$\left[-\frac{d^2}{dx^2} + V(x) + \varepsilon \mathbf{1}, \ a(x, \varepsilon) \frac{d}{dx} + b(x, \varepsilon) \right]$$
$$= r(x, \varepsilon) \left(-\frac{d^2}{dx^2} + V(x) + \varepsilon \mathbf{1} \right).$$

This gives a system of determining equations for the coefficients a, b which are Nth order polynomials in ε

$$a''(x, \varepsilon) + 2b'(x, \varepsilon) = 0,$$
$$b''(x, \varepsilon) + a(x, \varepsilon)V'(x) + 2a'(x, \varepsilon)(V(x) + \varepsilon) = 0,$$

where primes denote differentiation with respect to x. Using the ε-dependence we find after integration the forms of the functions $b_i(x)$

$$b_i(x) = -\frac{1}{2}a_i'(x) + B_i, \quad i = 0, 1, \ldots, N \qquad (22)$$

and $N+2$ recurrence relations for $a_j(x)$ depending on $V(x)$ and its derivatives

$$a_N(x) = 1,$$
$$a_{j-1}'(x) = -\frac{1}{4}a_j'''(x) - V(x)a_j'(x) - \frac{1}{2}V'(y)a_j(y), \qquad (23)$$

8

where B_j are integration constants, $a_j(x) \stackrel{\text{def}}{=} 0$ for $j = -1$, $j = N, N-1, \ldots, 0$. The set of equations (23) can be considered as differential equations for $V(x)$.

The first $N + 1$ relations of (23) are solved by subsequent integrations yielding the expressions for the functions $a_0(x), \ldots, a_{N-1}(x)$ via the function $V(x)$ and its derivatives. Substituting these results into the last equation for $j = 0$ we arrive at an $(2N + 1)$th order nonlinear differential equation for $V(x)$.

To reveal the structure of (23) we introduce new functions $\mathcal{U}_0(x), \mathcal{U}_1(x), \ldots$ by the following recurrence relation:

$$\mathcal{U}_j(x) = \mathcal{P}\,\mathcal{U}_{j-1}, \quad \mathcal{U}_{-1} \equiv 1,$$

$$\mathcal{P} = -\frac{1}{4}\frac{d^2}{dx^2} - V(x) + \frac{1}{2}\left(\frac{d}{dx}\right)^{-1} V'(x), \tag{24}$$

where $j = 0, 1, \ldots$ and $\mathcal{U}_{-1}(x) \stackrel{\text{def}}{=} 1$. Note that $\mathcal{P}$ is the first recursive (integro-differential) operator for the KdV equation (see, e.g. [7]–[9]). The action of $\mathcal{P}$ on some initial conserved density $\mathcal{U}_0 = -\frac{1}{2}V(x)$ yields the whole hierarchy of the conserved densities $\mathcal{U}_1, \mathcal{U}_2, \ldots$.

The essential point of the proof is that we can solve (23) for $j = N, N - 1, \ldots, 1$ in terms of $\mathcal{U}_j(x)$

$$a_{N-j}(x) = \mathcal{U}_{j-1} + \sum_{k=1}^{j-1} C_{N-k}\mathcal{U}_{j-k-1}(x) + C_{N-j}, \quad j = 1, \ldots, N, \tag{25}$$

where $C_0, \ldots, C_{N-1}$ are integration constants independent of constants B_j, $(j = 0, 1, \ldots, N)$.

As $a_{-1}(x) \stackrel{\text{def}}{=} 0$, the equation for $j = 0$ can be rewritten as (D_x denotes $\frac{d}{dx}$)

$$D_x \circ \mathcal{P}\, a_0(x) = 0$$

or, equivalently,

$$D_x \circ \left(\sum_{j=0}^{N-1} C_j\, \mathcal{P}^j + \mathcal{P}^N\right)\mathcal{U}_0 = 0, \tag{26}$$

where $\mathcal{U}_0 = -\frac{1}{2}V$, as starting point of the recursion.

The solutions of (26) generate all the solutions of (22), (23) which are differential equations for those $V(x)$ that allow for $H + \varepsilon\mathbf{1}$ a Lie symmetry with respect to a first-order operator (19).

In what follows we will show that (26) coincides with the stationary higher KdV equation. Using the operator identity

$$D_x \circ \mathcal{P}^j \equiv \left(D_x \circ \mathcal{P} \circ D_x^{-1}\right)^j \circ D_x$$

we relate the integro-differential operator $\mathcal{P}$ with the second recursive operator for the KdV equation $\mathcal{R}$,

$$\mathcal{R} = D_x \circ \mathcal{P} \circ D_x^{-1} = -\frac{1}{4}D_x^2 - V - \frac{1}{2}V'D_x^{-1}.$$

With the operator $\mathcal{R}$ we can represent equation (26) in the following form:

$$\left(\sum_{j=0}^{N-1} C_j \mathcal{R}^j + \mathcal{R}^N\right) \circ D_x \mathcal{U}_0 = 0, \quad \mathcal{U}_0 = -\frac{1}{2}V'(x).$$

Taking into account that $D_x \mathcal{U}_0 = -\frac{1}{2}V' = F_0$ and that $F_j = \mathcal{R}^j F_0$ we get finally (21). Thus the Schrödinger equation (18) admits a Lie symmetry of the form (19) if and only if the potential $V(x)$ is a solution of equation (21). $\triangleright$

Theorem 2 *The Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$ admits an n-th symmetry operator $Q \in \mathcal{L}$ with $[Q, H] = \kappa H$ if and only if the function $V(x)$ satisfies nonlinear differential equation*

$$\kappa\left(\frac{x}{2}V' + V\right) + \sum_{j=0}^{N-1} C_j F_j + F_N = 0 \tag{27}$$

with $N = [\frac{n-1}{2}]$, $n \geq 1$ and with $F_j = \mathcal{R}^j F_0$ under $\mathcal{F}_0 = -\frac{1}{2}V'$; C_i are some constants.

Proof. Computing the commutator on the left-hand side of (16) and equating coefficients of the operators $\frac{d^{n+1}}{dx^{n+1}}$ we have $q_n = \mathrm{const}$. Consequently, without loosing generality we may choose $q_n = 1$ and look for a symmetry operator Q in the form (20). Furthermore, as the operator Q belongs to $\mathcal{L}$, n is odd and, consequently, can be represented as $n = 2N + 1$ with some $N \in \mathbf{N}$.

With this remark (16) reads as

$$-\sum_{j=0}^{n-1} \begin{pmatrix} j \\ n \end{pmatrix} V^{(n-j)} \frac{d^j}{dx^j} - \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} q_i \begin{pmatrix} j \\ i \end{pmatrix} V^{(i-j)} \frac{d^j}{dx^j}$$
$$-\sum_{j=0}^{n-1} (2q'_j \frac{d}{dx} + q''_j) \frac{d^j}{dx^j} = \kappa \left(\frac{d^2}{dx^2} - V(x) \right). \tag{28}$$

Comparing coefficients in front of the linearly-independent operators $\frac{d^j}{dx^j}$, $(j = 1, \ldots n)$ yields n recurrence integro-differential relations for the coefficients $q_i(x, V(x), \kappa)$, $(i = 0, 1, \ldots, n-1)$ in the operator Q

$$q_{n-1}(x) = C_{n-1},$$
$$q_{j-1}(x) = -\frac{1}{2}\left(q'_j(x) + \begin{pmatrix} j \\ n \end{pmatrix} V^{(n-j-1)}(x) + \sum_{i=j+1}^{n-1} \begin{pmatrix} j \\ i \end{pmatrix} \right. \tag{29}$$
$$\left. \times \int^x q_i(y) V^{(i-j)}(y) dy + \kappa \delta_{j2} x \right) + C_{j-1},$$

where

$$C_j = \begin{cases} \tilde{C}_k, & j = 2k+1, \\ 0, & j = 2k, \end{cases}$$

$\tilde{C}_k$ are arbitrary constants.

Collecting the terms without derivative $\frac{d}{dx}$ in (28) we get an equation for $V(x)$ of the type

$$G(V, \kappa) \equiv q''_0 + V^{(n)} + \sum_{j=1}^{n-1} q_j V^{(j)} + \kappa V = 0. \tag{30}$$

Now we apply Theorem 1. For $\kappa = 0$ we know that the equation for $V(x)$ is given by (21). Hence

$$G(V, 0) \equiv G(V) = \sum_{j=0}^{N-1} C_j F_j + F_N, \quad N \in \mathbf{N} \tag{31}$$

holds with some constant $C_0, C_1, \ldots, C_{N-1}$.

11

On the other hand, an analysis of relations (29), where κ appears for $j = 2$ only, and (30) yields that $V(x) \in \mathcal{F}_L^\kappa$, where

$$\mathcal{F}_L^\kappa = \left\{ V(x) \mid G(V, \kappa) \equiv G(V) + \kappa \left(\frac{x}{2} V' + V \right) = 0 \right\}. \tag{32}$$

Combining the relations (31) and (32) we arrive at (27). $\triangleright$

This concludes the discussion of high order Lie symmetry of the Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$.

C. Relation to spectrum generating algebras

The Lie symmetry $[Q, H] = \kappa H$ is related to the spectrum generated algebra $so(2, 1)$ of H through (17), where $\kappa = -1$. The realization $R(g_2)$ of $g_2 \in so(2, 1)$ is given by Q of the form (14), (29) through solutions of (27) with $\kappa = -1$. As $R(g_2)$ is explicitly known, we can insert $R(g_2)$ and the differential operators $R(g_1)$, $R(g_3)$ into the commutation relations of the algebra $so(2, 1)$ and thus find the latter (for further details, see [1, 2]).

With the results obtained in III.B we can elucidate the peculiar features of the nonlinear differential equation (9) mentioned in Section II. The fact that the potentials $V(x)$ are identical for $n = 2N + 1$ and $n = 2N + 2$ is explained as follows. The coefficient in front of the highest power of the symmetry operator Q is equal to 1. Utilizing this property we can cancel this coefficient in the $(2N + 2)$th order symmetry operator Q by subtracting from it the trivial symmetry H^{N+1} thus getting a $(2N + 1)$th order symmetry operator $\tilde{Q}$. Evidently, the latter is admitted by the same Schrödinger equation which is the same as what was claimed. The stronger statement that the equations for $V(x)$ are identical for $n = 4k, 4k + 1, 4k + 2, 4k + 3, \ k = 1, 2, \ldots$ is valid for SGA appearing in [1, 2]. The way of constructing of the equation for $V(x)$ used while proving Lemma 1, makes it also evident, why this equation with some fixed $n = n_1$ contains all the terms of an equation for $V(x)$ under $n = n_2 < n_1$. Indeed, the equation for $n = n_1$ is obtained from one for $n_1 - 1$ with the action of the recursive operator $\mathcal{R}$ and the latter transforms a term F_i into F_{i+1} and, what is more, $\mathcal{R} \, 0 = \text{const}$.

D. Integrability

Hamiltonians (10) admitting $so(2,1)$ spectrum generating algebra have a further useful property, they are *integrable* in the sense that the corresponding Schrödinger equation (11) can be integrated by quadratures. This is so because (11) admits a first-order Lie symmetry of the form $X = \xi(x)\frac{d}{dx} + \eta(x)$ with

$$-\eta'' + 2V\xi' + V'\xi = 0, \quad 2\eta' + \xi'' = 0. \tag{33}$$

Hence we can apply for integration of equation (11) the classical method (see, e.g., [12]) based on its Lie symmetry. The first integral for system (33) is given by the following formula:

$$-\eta'\xi + V\xi^2 - \frac{1}{4} = \alpha \equiv \text{const.}$$

Depending on the sign of α the general solution of the equation (11) reads

$$\psi(x) = \sqrt{\xi(x)} \begin{cases} C_1 f(x) + C_2, & \alpha = 0 \\ C_1 \cos a f(x) + C_2 \sin a f(x), & \alpha = a^2 > 0, \\ C_1 \cosh a f(x) + C_2 \sinh a f(x), & \alpha = -a^2 < 0, \end{cases}$$

where

$$f(x) = \int \frac{dx}{\xi(x)}.$$

Now inserting the explicit expressions for $\xi(x), \eta(x)$ into the above formulae yields the general solution of (11) provided the function $V(x)$ fulfill an equation of the form (9).

IV. Concluding Remarks

Given a physical observable quantized through linear differential operator A in $L^2(\mathbf{R}^d, dx^d)$, e.g., the Hamiltonian H as in (1) and $d = 1$, a spectrum generating algebra for A is specified through a Lie algebra L having the dimension d (say, $so(2,1)$) with generators g_i, a realization R of L through differential operators of the order n and $H = \sum_{j=1}^p \alpha_j R(g_j)$. A high order Lie symmetry for the linear operator A is defined through finite order differential operators Q, P with $[Q, A] = PA$, e.g., $A = H$ and $P = \kappa$. Both SGA and high order Lie symmetry are different methods with different mathematical

structures. They model a symmetry of A. We have shown that for the Hamiltonian (1) $so(2,1)$-SGA and the Lie symmetry with $\kappa = -1$ are directly related via (17). However, Lie symmetry is more general than $so(2,1)$-SGA symmetry (see Section III.C). The interesting result is a connection to the stationary KdV hierarchy. It is understandable that for the singular case of $so(2,1)$, which reflects the light cone case, a family of non-linear differential equations for $V(x)$ appears. But it is, as we already mentioned, surprising that this family is the stationary KdV hierarchy, a mathematical object not connected directly to a symmetry concept of observables. We suspect that the KdV hierarchy is somehow encoded in the geometry of $so(2,1)$ and its realizations. An investigation of Hamiltonians of the type (1) with $d > 1$, their Lie symmetry in the above sense and L-SGA with non-compact L, $p > 3$, seems to be appropriate.

Acknowledgments

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